Variational Inequalities in Economics

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Variational Inequalities

Given a function $F : \Re^n \rightarrow \Re^n$ and convex set $C \subseteq \Re^n$

- Find $x^* \in C$ such that

$$F(x^*)^T (x - x^*) \geq 0 \quad \forall x \in C$$

- Find $x^* \in C$ to satisfy the generalized equation

$$0 \in F(x^*) + N_C(x^*)$$

- Find $z^* \in \Re^n$ to satisfy the nonsmooth equation

$$F(\Pi_C(z^*)) + z^* - \Pi_C(z^*) = 0$$

Special Cases

- Nonlinear equations ($C = \Re^n$)

$$F(x) = 0$$

- Nonlinear complementarity ($C = \Re^n_+$)

$$0 \leq x \quad \perp \quad F(x) \geq 0$$

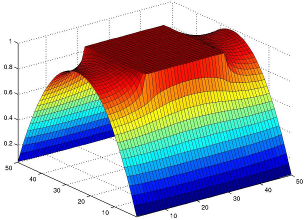
- Mixed complementarity ($C = [\ell, u]$)

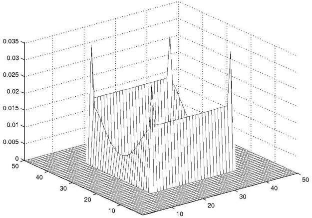
$$\ell \leq x \leq u \quad \perp \quad F(x)$$

  - If $\ell_i = x_i^*$, then $F_i(x^*) \geq 0$
  - If $\ell_i < x_i^* < u_i$, then $F_i(x^*) = 0$
  - If $x_i^* = u_i$, then $F_i(x^*) \leq 0$

Obstacle Problem

$$\min\left\{\int_{\mathcal{D}}\sqrt{1+\|\nabla v(x)\|^2}\,dx : v \geq v_L\right\}$$





Number of active constraints depends on the height of the obstacle. The solution $v \notin C^1$. Almost all multipliers are zero.

Grad-Shafranov Equation

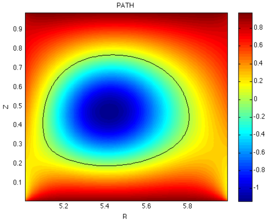
- Original problem

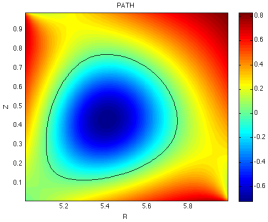
$$
\begin{array}{ll}
\Delta^* \psi + (\Lambda^2 r^2 + M)\psi = 0 & \text{if } \psi < 0 \\
\Delta^* \psi = 0 & \text{if } \psi > 0
\end{array}
$$

where $\Delta^* = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{1}{r^2}\frac{\partial}{\partial r}$

- Reformulation

$$0 \le \Delta^* \psi \quad \perp \quad \Delta^* \psi + (\Lambda^2 r^2 + M)\psi \ge 0$$

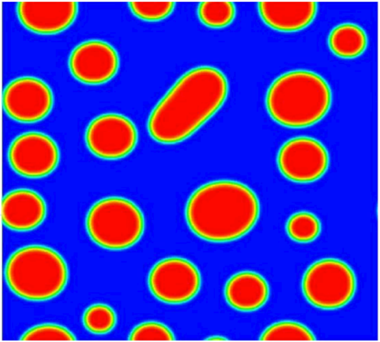




with J. Lee, L. Wang, M. Anitescu, L. McInnes, and B. Smith

Cahn-Hilliard Equation

Void formation in irradiated materials



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Some Properties

- Physical applications typically have unique solutions
- Free boundaries cause nonsmooth solutions
- Perturbation results are applicable
- Validation with manufactured solutions might be difficult
- Uncertainty propagation might be difficult

Nash Games

- Non-cooperative game played by n individuals
    - Each player selects a strategy to optimize their objective
    - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible

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x^* \in \begin{cases} \underset{x \geq 0}{\arg\min} & f_1(x, y^*) \\ \text{subject to } & c_1(x) \leq 0 \end{cases}
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y^* \in \begin{cases} \underset{y \geq 0}{\arg\min} & f_2(x^*, y) \\ \text{subject to } & c_2(y) \leq 0 \end{cases}
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\end{aligned}
$$

- Many applications in economics
  - Bimatrix games
  - Cournot duopoly models
  - General equilibrium models
  - Arrow-Debreau models

Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each y
  - $f_2(x, \cdot)$ is convex for each x
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$
\begin{array}{ll} \min\limits_{x \geq 0} & f_1(x, y^*) \\ \text{subject to } c_1(x) \leq 0 & \end{array}
\Leftrightarrow
\begin{array}{l} 0 \leq x \ \perp \nabla_x f_1(x, y^*) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\ 0 \leq \lambda_1 \perp -c_1(x) \geq 0 \end{array}
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$$\begin{array}{ll} \min\limits_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{array} \Leftrightarrow \begin{array}{l} 0 \leq y \;\perp \nabla_y f_2(x^*, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\ 0 \leq \lambda_2 \perp -c_2(y) \geq 0 \end{array}$$

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$$\begin{aligned}
0 \leq x &\perp \nabla_x f_1(x,y) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\
0 \leq y &\perp \nabla_y f_2(x,y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\
0 \leq \lambda_1 &\perp -c_1(y) \geq 0 \\
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\end{aligned}$$

- Nonlinear complementarity problem
  - Square system – number of variables and constraints the same
  - Each solution is an equilibrium for the Nash game

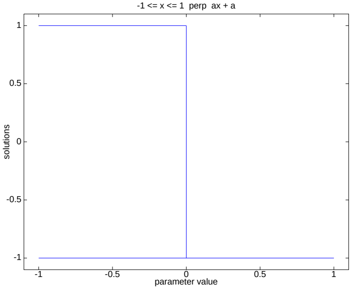
Properties

- Problems can have many solutions
  - No solution
  - Finite number of distinct solutions
  - Union of convex sets
- Sometime you want to know all solutions
- Free boundaries cause nonsmooth solutions
- Perturb parameters and solution characteristics change
- Validation with manufactured solutions difficult
  - Need initial set of solutions
  - Manufactured problem may have more solutions
- Quality of interest difficult to define
  - Function from union of convex sets to $\Re$

"Simplest" Example

$$-1 \le x \le 1 \quad \perp ax + a$$

where a is unknown in the range $[-1, 1]$.



General Equilibrium Models – Producers

- Producers maximize profit subject to production constraints

$$\bar{x} \in \max_{x \in X} \bar{p}^T x$$

  - Prices are given
  - Choose optimal quantities
  - Constant elasticity of substitution constraints
    - Cobb-Douglas and Leontief special cases
    - Nesting based on sector

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  - Account for transportation costs

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- Importers are treated as producers
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- Data available by sector
  - Expenditures
  - Revenues
  - Taxes

General Equilibrium Models – Consumers

- Consumers maximize utility subject to budget

$$\bar{y} \max_{y \in Y(\bar{x},\bar{p})} g(y)$$

  - Receive dividends from producers
  - Receive tax revenue from government
  - Use revenue to buy goods and services
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General Equilibrium Models – Markets

- Markets set commodity prices so that supply equals demand

$$0 \le \bar{p} \;\perp\; \bar{x} + \bar{y} \ge 0$$

  - Supply and demand are given
  - If supply exceeds demand then the price is zero
  - If supply equals demand then the price can be positive

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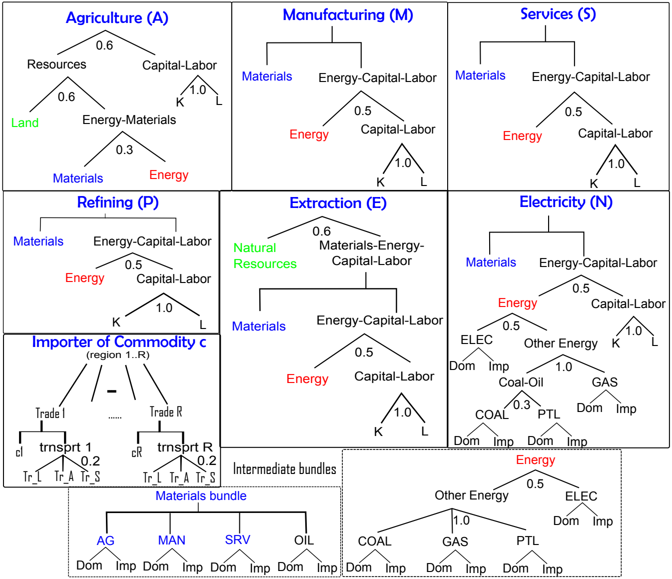
  - If supply equals demand then the price can be positive
- Collection of optimization problems and complementarity constraints

$$\begin{aligned} \bar{x} &\in \max_{x \in X} \bar{p}^T x \\ \bar{y} &\in \max_{y \in Y(\bar{x},\bar{p})} g(y) \\ 0 \le \bar{p} &\perp \bar{x} + \bar{y} \ge 0. \end{aligned}$$

Nested Functions



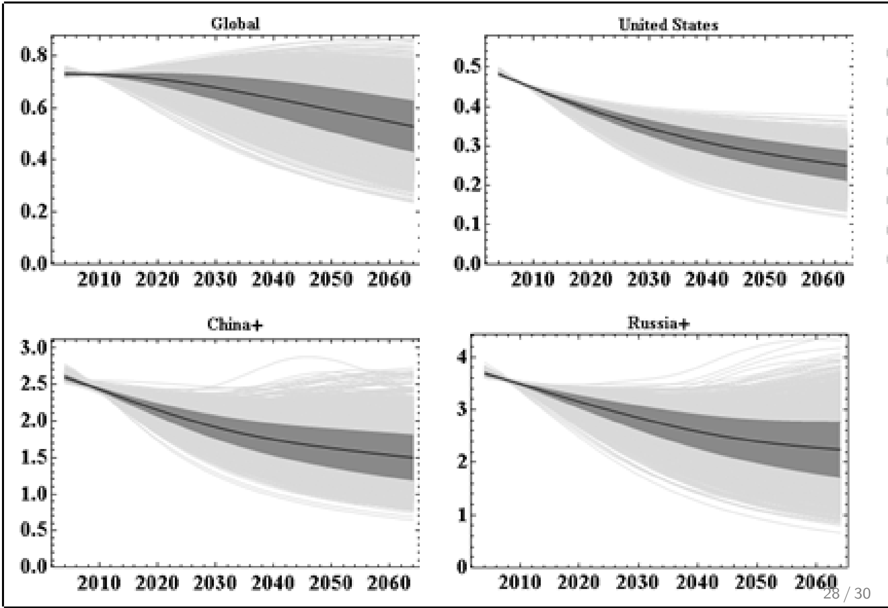
Estimation and Calibration

- Estimation
  - Compute elasticities and standard errors from data
  - Discrete choices in tree structure and standard errors
  - Compute dynamic trajectories from data and extrapolation
- Calibration
  - Choose share parameters to clear markets to replicate base year data
- Limited "validation"
  - Train with 2005–2010 data
  - Hindcast to 1990–2005
  - Compare to historical data

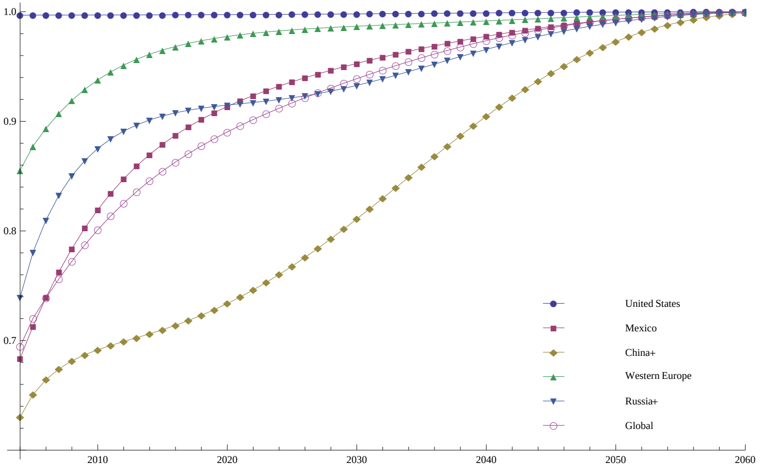
Uncertainty Quantification

- Many unknowns per region
  - Estimated elasticity parameters
  - Tree structure of the functions
  - Base-year expenditure data
  - Dynamic trajectories
  - Model type
- Some parameters may be correlated, but its unclear
- Dimensionality reduction may not be possible
- We use simple Monte Carlo simulation

Some Results – Carbon Intensity



Some Results – GDP



Conclusions

- Variational inequalities in economics have interesting properties
  - Nonsmooth solutions
  - Multiplicity of solutions
  - Sharp transitions in solution types
  - Many sources of (irreducible) uncertainty
  - Limited opportunities to perform experiments